# ANNIHILATORS OF IDEALS OF EXTERIOR ALGEBRAS

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Abstract. The Orlik-Solomon algebra $A$ of a matroid is isomorphic to the quotient of an exterior algebra $E$ by a defining ideal $I$. We find an explicit presentation of the annihilator ideal of $I$ or, equivalently, the $E$-module dual to $A$. As an application of that we provide a necessary, combinatorial condition for the algebra $A$ to be quadratic. We show that this is stronger than matroid being line-closed thereby resolving (negatively) a conjecture by Falk. We also show that our condition is not sufficient for the quadraticity.

## 1. Introduction

This paper concerns ideals of exterior algebras, more precisely the ideals related to the Orlik-Solomon (OS) algebra of a matroid. OS algebras appeared first from theorems of Brieskorn and Orlik-Solomon as the cohomology algebras of complements of arrangements of hyperplanes in a complex linear space. The Orlik-Solomon theorem showed in particular that the OS algebra of an arrangement is defined by the underlying matroid and the definition is valid for an arbitrary simple matroid (not necessarily representable over $\mathbb{C}$). Such a matroid $M$ on the set $[n] = \{1, 2, \dots n\}$ defines the exterior algebra $E$ with $n$ generators and its graded ideal $I$ (the OS ideal of $M$). Then the OS algebra of $M$ is the graded algebra $A = E/I$.

The OS algebras have been extensively studied during the last 20 years from different points of view and for different applications (see books [12, 13] and surveys [8, 17]). We recall here two open problems. The older one is to find a condition on $M$ equivalent for $A$ being quadratic. This is important because the quadraticity of $A$ is first step to $A$ being Koszul. The latter property is equivalent (for matroid representable over $\mathbb{C}$) to the arrangement complement being rational $K[\pi, 1]$ whence it relates to other topological properties (see [14, 7]). Several years ago, Falk proved that the well-known property of $M$ being line-closed is necessary for $A$ being quadratic and conjectured that it is also sufficient.

The other problem is a very recent one. In [6], the problem of resolving $A$ as a graded $E$-module is considered. While very little is known about free

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resolutions of $A$, the main theorem of [6] proves that its minimal injective resolution is linear and computes the betti numbers of $A$. However, it leaves open the apparently hard problem of describing this resolution explicitly.

In the present paper, we make progress on both problems by studying the annihilator $I^0$ of $I$. Since $I^0$ is isomorphic to the dual module to $A$, constructing an injective resolution of $A$ is equivalent to constructing a free resolution of $I^0$. We explicitly find the two initial terms of this resolution. First, we exhibit generators of $I^0$ (in fact a Gröbner basis). Although we do this directly one could also use the basis of the flag space from [16]. Then we find the generating relations among these generators. The description of these relations is messier; although these relations can be obtained by a deformation of the simple relations for the initial ideal of $I^0$, this deformation needs to be chosen carefully.

To study the quadraticity of $A$ one notices that it is equivalent to the equality $I^0 = J^0$, where $J$ is the ideal of $E$ generated by the degree 2 component of $I$. This equality allows us to define a property of $M$ which we call *3-independent* that implies line-closed and is still necessary for the quadraticity. This property relates to the question of into how many parts $[n]$ can be broken so that each circuit has at least two points common with some of the parts. We give an example (Example 4.5) of a line-closed matroid that is not 3-independent, disproving Falk's conjecture. We find that while both ideals $I$ and $J$ are generated by pure (decomposable) elements, their annihilators have different properties in general. The ideal $I^0$ is also generated by pure elements (in particular the generators mentioned above are pure); $J^0$ is not necessarily so (see Example 4.6). When this happens our criterion may not work. In particular the matroid of Example 4.6 is 3-independent but $A$ is not quadratic.

The setup of the paper is as follows. In Section 2, we study pure ideals (i.e., ideals generated by pure elements), in particular ideals generated by the degree $p$ component of the OS ideal of $M$. In Section 3, we prove that $I^0$ is pure by exhibiting its pure generators. We also prove that these generators form a Gröbner basis and exhibit a basis of every homogeneous component of this ideal. In Section 4, we define $p$-independent matroids and establish relations among the criteria of being 3-independent, line-closed, and quadratic. In Section 5, we give a basis of the relation space among generators of $I^0$.

## 2. Pure ideals

Let $R$ be an arbitrary commutative ring with 1, $V$ a free module over $R$ of finite rank, and $E = \Lambda V$ the exterior algebra of $V$. We view $E$ as a graded $R$-algebra with the standard grading $E_p = \Lambda^p V$. In particular $E_1 = V$.

Recall that an element $a \in E$ is called *pure* (decomposable) if it is the product of elements of degree 1. Unless $a \in V$, the linear factors of $a$ are

not uniquely defined. We call an ideal $I$ of $E$ *pure* if it is generated by a set of pure elements.

The following classes of pure ideals constitute the main characters of the paper. Let $M$ be a simple (no loops or nontrivial parallel classes) matroid on the set $[n] = \{1, 2, \dots, n\}$. Let $V$ be the $R$-module with a basis $\{e_1, e_2, \dots, e_n\}$. Then there is the natural homogeneous basis of $E$ consisted of all the monomials $e_S = e_{i_1} e_{i_2} \cdots e_{i_p}$ one for each subset $S = \{i_1, \dots, i_p\} \subset [n]$ (in sections 1-4, we always assume that $i_1 < i_2 < \cdots < i_p$). This basis allows one to define the structure of a differential graded algebra on $E$. The differential $\partial : E \to E$ of degree -1 is defined via $\partial e_i = 1$ for every $i = 1, 2, \dots, n$ and the Leibniz rule

$$\partial(ab) = (\partial a)b + (-1)^{\deg a} \partial b$$

for every homogeneous $a$ and every $b \in E$. In this notation, the Orlik-Solomon (OS) ideal of $M$ is the ideal $I(M)$ of $E$ generated by $\partial e_S$ for every *dependent* set $S$ of $M$. Clearly $I(M)$ is homogeneous, $I(M) = \oplus_{p=2}^{n} I_p(M)$ where $I_p(M) = I(M) \cap E_p$. Denote by $J(p, M)$ the ideal of $E$ generated by $I_r(M)$ with $r \leq p$. Notice that $J(p, M) \subset J(p+1, M)$, $J(p, M) = 0$ for $p < 2$, and $J(\ell, M) = I(M)$ where $\ell$ is the rank of $M$. A straightforward check shows that for every $S = \{i_1, \dots, i_p\} \subset [n]$ we have

$$\partial e_S = (e_{i_2} - e_{i_1})(e_{i_3} - e_{i_1}) \cdots (e_{i_p} - e_{i_1}). \tag{2.1}$$

Thus all the ideals $J(p, M)$ are pure ideals.

To each ideal $I$ of $E$ we relate its annihilator $I^0$ that is the ideal $I^0 = \{a \in E | ab = 0, \text{ for every } b \in I\}$. Clearly $I^0$ is homogeneous if $I$ is. There is a natural isomorphism of graded $E$-modules $\mathrm{Hom}_E(E/I, E) \cong I^0$ assigning to $\phi : E/I \to E$ the element $\phi(1)$. Also fixing a basis in $V = E_1$ identifies $E_n$ with $R_n$ that is the trivial $E$-module whose grading is concentrated in degree $n$. Then the product in $E$ defines for every $p = 1, 2, \dots, n$ the nondegenerate $R$-bilinear pairing $E_p \times E_{n-p} \to R_n$. These pairings define the isomorphism of $E$-modules $(E/I)^* = \mathrm{Hom}_R(E/I, R_n) \cong I^0$ that preserves the degree. In particular we have the canonical isomorphism of free $R$-modules $I_p^0 \cong (E/I)_{n-p}$. This implies also that $(I^0)^0 = I$.

Let us fix a simple matroid $M$ on $[n]$ of an arbitrary rank $\ell$ and put $I = I(M)$. Extending Definition 2.64 from [12] we say that a partition $\pi$ of $[n]$ is *$p$-independent* for some $p$, $3 \leq p \leq \ell + 1$, if any choice of one element from each of not more than $p$ arbitrary elements of $\pi$ forms an independent set of $M$. It is clear that $(\ell+1)$-independent partition can have at most $\ell$ elements. Following [12] we call $(\ell+1)$-independent partition *independent.*

For any partition $\pi$ of $[n]$, call its parts $A_1, A_2, \dots, A_k$ (in an arbitrary order). Let $\sigma \in \Sigma_n$ be the permutation (the shuffle of $A_1$ through $A_k$) that puts the elements of $A_i$ before those of $A_{i+1}$ for each $i \leq k$ preserving the order inside each $A_i$ induced from $[n]$. Then using this order on $A_i$ put

$$z(\pi) = \mathrm{sign}(\sigma) \partial(e_{A_1}) \partial(e_{A_2}) \cdots \partial(e_{A_k}). \tag{2.2}$$

Clearly $z(\pi) \in E_{n-|\pi|}$.

**Lemma 2.1.** *If a partition $\pi$ of $[n]$ is $p$-independent then*

$$z(\pi) \in J(p-1, M)^0_{n-|\pi|}.$$

*Proof.* Let $S \subset [n]$ be a dependent set of $M$ and $|S| \le p$. It suffices to prove that $z = z(\pi)$ annihilates $\partial e_S$. Since $\pi$ is $p$-independent there exists at least one $A \in \pi$ such that $A \cap S$ contains at least two elements, say it contains $r$ and $s$. Then both $z$ and $\partial e_S$ are divisible by $e_r - e_s$ whence $z\partial e_S = 0$. This completes the proof. □

For $p < \ell + 1$ the elements $z(\pi)$ for $p$-independent partitions do not necessarily generate $J(p-1, M)^0$. Moreover this ideal may not have any pure elements in its minimal degree component, in particular it may not be a pure ideal. One of the celebrated Hilbert–Cohn-Vossen matroids, namely $(9_3)_2$ (see [11]), provides such an example when $\ell = p = 3$: see Example 4.6. The results below show that the OS ideals of matroids are special in this respect.

## 3. OS-ideals

In this section, we focus our attention on the OS ideal $I = J(\ell, M) = I(M)$. Our goal is to exhibit a Gröbner basis of this ideal and a generating set consisting of certain $z(\pi)$.

It turns out that it suffices to consider only independent partitions of the following special type. Let $F = (X_0 = \emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_\ell = [n])$ be a maximal flag of flats of $M$ (or a maximal chain in the lattice $L$ of all the flats of $M$). Put $S_i = X_i \setminus X_{i-1}$ $(i = 1, 2, \ldots, \ell)$ and denote by $\pi(F)$ the ordered partition $\{S_1, \ldots, S_\ell\}$.

**Lemma 3.1.** *For every maximal flag $F$ the partition $\pi(F)$ is independent.*

*Proof.* In the above notation for $F$ let $i_j \in S_j$ for every $j$, $1 \le j \le \ell$. It suffices to prove that the set $T = \{i_1, \ldots, i_\ell\}$ is independent. If $T$ is dependent and $T_p = \{i_1, \ldots, i_p\}$ is its inclusion minimal dependent subset then $T_p \subset X_{p-1}$, which contradicts the choice of $i_p$. The contradiction completes the proof. □

To simplify the notation put $z(F) = z(\pi(F))$ for every maximal flag $F$. The previous two lemmas imply that $z(F) \in I^0$. Now we reduce the class of the partitions further. Recall that a *broken circuit* of $M$ is a circuit (i.e. an inclusion minimal dependent set) with its smallest element deleted. Then a set $T \subset [n]$ is called **nbc** if there is no broken circuits lying in it. We denote the collection of all **nbc**-sets of cardinality $p$ by $\mathbf{nbc}_p$. Clearly an **nbc**-set is independent. If $T = \{i_1, \ldots, i_p\}$ (recall that $i_1 < i_2 < \cdots < i_p$) then it is **nbc** if and only if $i_r = \min cl(\{i_r, \ldots, i_p\})$ for each $r$, $1 \le r \le p$. Here by $cl(S)$ we mean the closure of $S$, i.e., the minimal flat of $M$ containing $S$.

It is well known that the set of monomials $e_T$ where $T$ is running through $\mathbf{nbc}_p$ projects to a basis of $(E/I)_p$ under the natural projection $E \to E/I$.

Now fix an ordered **nbc**-set $T = (i_1, i_2, \cdots, i_\ell)$ of (maximal) length $\ell$ and define the maximal flag $\mathcal{F}(T) = (X_0 = \emptyset \subset X_1 \subset \cdots \subset X_\ell = [n])$ via $X_p = cl(\{i_{\ell-p+1}, \ldots, i_\ell\})$. To simplify the notation put $z(T) = z(\mathcal{F}(T))$.

**Lemma 3.2.** *Consider the lexicographic ordering on the subsets of a cardinality $p$ of $[n]$ and the respective ordering on the monomials in $E_p$. Then for each* **nbc***-set of length $\ell$ the largest (*leading*) monomial of $z(T)$ is $e_{\overline{T}}$ where $\overline{T} = [n] \setminus T$.*

*Proof.* Put $F = \mathcal{F}(T)$ and denote the respective partition of $[n]$ by $\pi$. For each $A \in \pi$ denote by $\nu(A)$ the minimal element of $A$. Then, by definition (2.2) of $z(T)$, the leading monomial of $z(T)$ is the product of the leading monomials of $\partial(e_A)$ for each $A \in \pi$. This equals $e_{A \setminus \{\nu(A)\}}$. Since $T$ is **nbc** we have $T = \{\nu(A) | A \in \pi\}$ which completes the proof. □

Put $Z = \{z(T) | T \in \mathbf{nbc}_\ell$. The above lemmas imply the main result of this section

**Theorem 3.3.** *The set $Z$ is a Gröbner basis of $I^0$ (with respect to the degree-lexicographic monomial ordering).*

*Proof.* For every set $A \subset E$, let $\mathrm{In}(A)$ denote the set of all leading (i.e. maximal in the degree-lexicographic ordering) monomials of elements of $A$. To prove the statement we need to show that $\mathrm{In}(I^0) = \langle \mathrm{In}(Z) \rangle$ where the right hand side is the set of all monomials divisible by some monomials from $\mathrm{In}(Z)$. Since $Z \subset I^0$ it suffices to prove the inclusion

$$\mathrm{In}(I^0) \subset \langle \mathrm{In}(Z) \rangle \,, \tag{3.1}$$

the opposite inclusion being obvious. It is clear also that

$$\mathrm{In}(I^0) \subset \overline{\mathrm{In}(I)} \tag{3.2}$$

where $\overline{\mathrm{In}(I)}$ consists of all monomials annihilating all elements of $\mathrm{In}(I)$. Recall that $\mathrm{In}(I)$ consists of all the monomials divisible by some monomials $e_U$ corresponding to broken circuits $U$. Thus $\overline{\mathrm{In}(I)}$ consists of monomials $e_T$ corresponding to the sets $T$ transversal to all the broken circuits, i.e., intersecting each one of them nontrivially. We will call these sets **tbc**. It is easy to see that the collection of **tbc**-sets consists of the complements to **nbc**-sets. In particular the inclusion minimal **tbc**-sets have $n - \ell$ elements. Thus

$$\overline{\mathrm{In}(I)} = \langle \mathrm{In}(Z) \rangle \,. \tag{3.3}$$

Now (3.2) and (3.3) imply (3.1), which concludes the proof. □

The following corollary is a routine application of Gröbner basis theory.

**Corollary 3.4.** *The set $Z$ is a generating set of $I^0$. In particular $I^0$ is a pure ideal.*

Corollary 3.4 implies a result for the OS ideal itself that may have an independent use. For each **nbc**-set $T$ of length $\ell$, denote by $I_T$ the linear ideal (i.e., generated in degree 1) generated by the factors of $z(T)$ (for instance, by $(e_i - e_{\nu(A)})$ for all $A \in \pi(T)$ and $i \in A \setminus \{\nu(A)\}$. It is a particular case of a well-known property of exterior algebras that $(I_T)^0 = Ez(T)$, where the right hand side is the principal ideal generated by $z(T)$.

**Corollary 3.5.** *The OS ideal $I$ is the intersection of linear ideals. More precisely*

$$I = \bigcap_{T \in \mathbf{nbc}_\ell} I_T.$$

*Proof.* Corollary 3.4 gives $I^0 = \sum_T Ez(T)$. Thus we have

$$I = (I^0)^0 = (\sum_T Ez(T))^0 = \bigcap_T (Ez(T))^0 = \bigcap_T I_T.$$

□

Theorem 3.3 allows us to exhibit a basis for each $(I^0)_p$ (as a free $R$-module). For that we need to define elements $z(T) \in I^0$ for $T \in \mathbf{nbc}_p$, $1 \le p \le \ell$. The flag $\mathcal{F}(T) = (X_0 \subset X_1 \subset \cdots \subset X_p)$ is defined exactly as for $p = \ell$ but it is no longer maximal. More precisely, its maximal element $X_p$ is not the maximal element of $L$. The sets $S_i = X_i \setminus X_{i-1}$ $(i = 1, \ldots, p)$ form a partition $\pi = \pi(\mathcal{F}(T))$ of $X_p$ that is still independent. Extend the definition of $z$ from (2.2) by setting

$$z(T) = z(\pi) = \operatorname{sign}(\sigma)\partial(e_{S_1})\partial(e_{S_2})\cdots\partial(e_{S_p})e_{[n]\setminus X_p} \tag{3.4}$$

where $\sigma$ is the shuffle of $X_p$ that puts $S_1$ through $S_p$ in the order of their indices and $[n] \setminus X_p$ after $S_p$. Notice that the definition coincides with (2.2) if $p = \ell$.

The properties of $z(T)$ for $|T| = \ell$ can be generalized to an arbitrary $p$.

**Lemma 3.6.** *For each **nbc**-set $T$, the leading monomial of $z(T)$ is $e_{\overline{T}}$.*

The proof is similar to proof of Lemma 3.2.

**Lemma 3.7.** *For each $T \in \mathbf{nbc}_p$ we have*

$$z(T) \in (I^0)_{n-p}.$$

*Proof.* Extend $\mathcal{F}(T)$ to a maximal flag $\tilde{F} = (X_0 \subset \cdots X_p \subset X_{p+1} \subset \cdots \subset X_\ell)$. By Lemmas 3.1 and 2.1, $z(\tilde{F}) \in I^0$. A straightforward check shows that $z(T) = \pm z(\tilde{F}) e_{\nu(X_{p+1}\setminus X_p)} \cdots e_{\nu(X_\ell \setminus X_{\ell-1})}$. Thus $z(T) \in I^0$ which completes the proof. □

Put $Z_p = \{z(T) | T \in \mathbf{nbc}_p\}$.

**Proposition 3.8.** *For every $p$, the set $Z_p$ form a basis of $(I^0)_{n-p}$.*

*Proof.* Since the $R$-module $(I^0)_{n-p}$ is isomorphic (noncanonically) to $(E/I)_p$, we see that the cardinality of $Z_p$ is correct. Thus it suffices to prove only that $Z_p$ generates $(I^0)_{n-p}$.

Let $e \in (I^0)_{n-p} \setminus \{0\}$. By Theorem 3.3, $\mathrm{In}(e) = \pm e_S e_{\overline{T}}$ for $T \in \mathbf{nbc}_\ell$. Since the set $U = [n] \setminus (S \cup \overline{T})$ is again $\mathbf{nbc}$, we can consider $z(U)$. Since $\mathrm{In}(z(U)) = e_U$, we have $e + cz_U \in (I^0)_{n-p}$ for some $c \in R$ and $\mathrm{In}(e + cz_U) < \mathrm{In}(e)$. Now the proof can be completed by induction on $\mathrm{In}(e)$. □

**Remarks**

1. In fact, the proof of Proposition 3.8 contains the proof that $Z_p$ is a Gröbner basis of the ideal $\bigcup_{k \geq n-p} (I^0)_k$.
2. The signs of $z(T)$ in the formula (3.4) is chosen so that
$$e_T z(T) = 1$$
for every $\mathbf{nbc}$-set $T$.
3. It is not hard to prove that the correspondence $F \mapsto z(F)$ induces a map $\mathrm{Fl}_p \to (I^0)_{n-p}$ where Fl is the graded flag module introduced by Schechtman and Varchenko in [16] (cf. Lemma 5.9 below). Then Proposition 3.8 can be deduced from the results of [16].

## 4. Quadratic OS-algebras

As in the previous section we fix a simple matroid $M$ on the set $[n] = \{1, 2, \dots, n\}$. Its Orlik-Solomon algebra (OS algebra) is $A = A(M) = E/I(M)$. This algebra receives the grading from $E$. In this section we study relations of annihilators of ideals of $E$ with conditions for the quadraticity of $A$. The latter property means that $I(M)$ is generated in degree 2 (equivalently $J(2, M) = I(M)$).

A condition sufficient for the quadraticity of $A$ was obtained by Falk in [7]. We recall it here. A subset $S \subset [n]$ is called *line-closed* (lcl) if it is closed with respect to 3-circuits (i.e., dependent sets of 3 elements). This means that if for a 3-circuit $C$ we have $|S \cap C| \geq 2$ then $C \subset S$. Clearly every flat of $M$ is lcl but the converse is false in general. If every lcl set is a flat then $M$ itself is called *line-closed* (lcl). Notice that if the rank of $M$ is 3 then $M$ is lcl if and only if the line-closure of any subset $S \subset [n]$ of rank 3 is the whole $[n]$.

**Theorem 4.1** (Falk [7])**.** *If $A(M)$ is quadratic then $M$ is lcl.*

An alternative proof of this theorem is given below.

Falk has conjectured that the converse of Theorem 4.1 is also true and the question was open for a while. The ideal annihilators allow us to solve this question negatively.

**Definition 4.2.** *Let $3 \leq p \leq \ell$. The matroid $M$ is* $p$-independent *if any $p$-independent partition of $[n]$ is independent.*

If rk$M = p = 3$ then the following rephrasing of the above definition is convenient. $M$ is 3-independent if and only if any graph on the set of points of $M$, having for each line of $M$ at least one edge lying on this line, has at most 3 components. The next theorem shows that being 3-independent is also necessary condition for the quadraticity.

**Theorem 4.3.** *If $A(M)$ is quadratic then $M$ is 3-independent.*

*Proof.* The condition of the theorem means that $J(2, M) = I(M)$ whence $(J(2, M))^0 = I^0$. Let $\pi$ be a 3-independent partition of $[n]$. By Lemma 2.1 $z(\pi) \in (J(2, M))^0$ whence $z(\pi) \in I^0$.

Assume that $\pi$ is not independent, i.e., there exists a dependent subset $S \subset [n]$ such that $|S \cap P| \leq 1$ for every $P \in \pi$. To find a contradiction it suffices to show that $z(\pi)\partial e_S \neq 0$. Indeed we have equality (up to a sign)

$$z(\pi)\partial e_S = \pm z(\bar{\pi})$$

where $\bar{\pi}$ is the partition obtained from $\pi$ by gluing together all the elements $P \in \pi$ such that $S \cap P \neq \emptyset$. Since for every partition $\rho$ we have $z(\rho) \neq 0$ we obtain the contradiction implying the result. □

As the next result we show that Falk's condition is implied by ours.

**Theorem 4.4.** *If $M$ is 3-independent than it is line-closed.*

*Proof.* Suppose $M$ is not lcl. To prove the theorem it suffices to exhibit a 3-independent partition having more than $\ell$ elements (whence not independent).

Let $S$ be a lcl subset of $[n]$ that is not a flat of $M$. Let $T$ be the closure of $S$. Denote by $r$ the common rank of $S$ and $T$ and include $T$ into a maximal flag $F$ of flats of $M$ between $T$ and $[n]$. Fixing notation, $F = (X_r = T \subset X_{r+1} \subset \cdots \subset X_\ell = [n])$. Since rank of $S$ is $r$ we can find a sequence $(S_0 = \emptyset \subset S_1 \subset \cdots \subset S_r = S)$ of $S$ such that rank of $S_i = i$. Now we can define the partition $\pi$ that consists of the following $\ell + 1$ elements: $S_p \setminus S_{p-1}$ for $p = 1, 2, \ldots, r$, $S' = T \setminus S$, and $X_i \setminus X_{i-1}$ for $i = r + 1, \ldots, \ell$.

We claim that $\pi$ is 3-independent. Let $X$ consist of 3 elements, one from each of 3 distinct elements of $\pi$. If at least one of these elements does not belong to $T$ then the independence of $X$ is obvious (cf. Lemma 2.1). The same is true if at least two of the elements belong to $S_{r-1}$. Thus the only interesting case is when $X = \{a, b, c\}$ where $a \in S_{r-1}, b \in Y_r = S \setminus S_{r-1}$, and $c \in S' = T \setminus S$. In this case, we have $|X \cap S| = 2$ and if $X$ is linearly dependent this contradicts the condition that $S$ is line-closed. Thus $\pi$ is 3-independent that concludes the proof. □

Notice that Theorems 4.3 and 4.4 immediately imply Theorem 4.1.

The following example shows that the converse to Theorem 4.4 is false, i.e., the independence condition is strictly stronger than the line-closure one.

**Example 4.5.** *Let $n = 8$ and $M$ the matroid of rank 3 shown in Figure 1(a). Its underlying set is $[8] = \{1, \ldots, 8\}$, and it has the following*

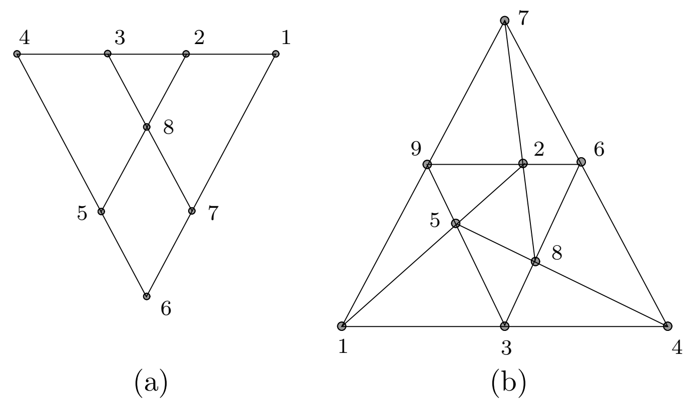



FIGURE 1. (a) $M$ of Example 4.5; (b) the $(9_3)_2$ matroid;

*maximal dependent sets of rank 2:*

$$\{\{1,2,3,4\},\{1,6,7\},\{2,5,8\},\{3,7,8\},\{4,5,6\}\}.$$

*By directly checking each independent set of rank* 3*, it is easy to see that the line-closure of each of them is* $[8]$*; that is,* $M$ *is line-closed. On the other hand, consider the partition*

$$\pi = \{\{1,3,7\},\{2,4,5\},\{6\},\{8\}\}.$$

*Since none of the dependent 3-sets intersect non-trivially with more than two elements of* $\pi$*, it is 3-independent. However, since* $\pi$ *has* 4 *elements, it is not independent. Hence* $M$ *is not 3-independent. Notice that the graph consisting of* 2 *triangles with the vertices* $1,3,7$ *and* $2,4,5$ *plus two isolated points* 6 *and* 8 *has an edge on each line of the matroid.*

The next example shows in turn that the converse to Theorem 4.3 is also false: if $M$ is 3-independent, $A(M)$ need not to be quadratic.

**Example 4.6.** *Consider the rank* 3 *matroid* $M$ *with nine points shown in Figure 1(b). This matroid first appeared in the book* [11] *by Hilbert and Cohn-Vossen, where it is called the* $(9_3)_2$ *configuration. Since* $M$ *is fixed we put* $J = J(2,M)$ *for the length of this example.*

*That* $A(M)$ *is not quadratic can be seen by the following simple computation (that applies in fact to the other two* $9_3$ *configurations from* [11]*). The ideal* $J$ *is generated by 9 elements from* $E_2$*. Since each of them is annihilated by a 2-dimensional subspace of* $E_1$ *we have*

$$\dim J_3 \leq 7 \times 9 = 63. \tag{4.1}$$

*On the other hand, the Hilbert series of* $A(M)$ *should be divisible by* $1+t$ *whence a simple computation gives* $H(A(M),t) = 1 + 9t + 27t^2 + 19t^3$*. This*

*implies*

$$\dim I_3 = \binom{9}{3} - 19 = 65. \tag{4.2}$$

*Thus* $J \neq I$.

*Proving that* $M$ *is 3-independent requires more work. Let us say for convenience that a subset* $T$ *of the vertices of* $M$ represent *a 3-circuit* $S$ *if* $|T \cap S| \geq 2$. *A useful observation about* $M$ *is that it has an automorphism of order 9 that can be written as a permutation* $\tau = [261594837]$ *in two-line notation. In particular, it is transitive on the vertices. Now we state the following for an arbitrary subset* $T$ *of the vertices of* $M$ *(it suffices to consider cases where* $1 \in T$*).*

(1) *If* $|T| = 3$ *then it represents at least one circuit.*

*Indeed the only vertices that do not represent a circuit with 1 are 6 and 8 and* $\{6, 8\}$ *represents a circuit.*

(2) *If* $|T| = 4$ *then it represents at least 2 and at most 5 circuits.*

*There are two cases. Suppose* $T$ *contains a circuit* $S$ *(one can assume* $S = \{1, 3, 4\}$ *or* $S = \{1, 2, 5\}$*). Then by inspection any other vertex represents another circuit with one of the vertices of* $S$. *The first statement follows. Clearly in this case* $T$ *can represent at most 4 circuits.*

*Now suppose* $T$ *does not contain a circuit. Since two distinct subsets of* $T$ *of cardinality* 3 *cannot represent the same circuit, again the first assertion must hold. On the other hand,* $M$ *viewed as a graph does not have a complete subgraph of with* 4 *vertices, and this implies the second assertion.*

(3) *If* $|T| = 5$ *then it represents at most* 7 *circuits. If* $|T| = 6$ *then this number is* 8.

*This follows from (1) and (the first assertion of) (2) by passing to the complements.*

*Now suppose* $\pi$ *is a* 3*-independent partition of the set of vertices of* $M$. *This means that the elements of* $\pi$ *represent each of the* 9 *circuits. Considering all the possibilities for the cardinalities of elements of* $\pi$ *if* $|\pi| = 4$, *i.e.* $\{6, 1, 1, 1\}$, $\{5, 2, 1, 1\}$, $\{4, 3, 1, 1\}$, $\{4, 2, 2, 1\}$, *we deduce from (the second statement of) (2) and (3) that the elements of* $\pi$ *can represent at most* 8 *elements. Thus* $|\pi| \leq 3$ *whence* $\pi$ *is independent. This shows that* $M$ *is* 3*-independent.*

*In fact using Macaulay 2* [10] *we can obtain a basis of* $(J^0)_5$ *and show that* $(J^0)_5$ *does not contain any pure elements (whence* $J^0$ *is not a pure ideal). To describe this basis let*

$$x = e_1e_2e_3e_6e_8 - e_1e_2e_4e_6e_9 + e_1e_2e_4e_8e_9 - e_1e_2e_6e_8e_9 - e_2e_3e_5e_6e_9,$$

$p = (1 - \tau)(1 + \tau^3 + \tau^6)x$, *and* $q = \tau p$. *Then* $p$ *and* $q$ *form a basis for* $(J^0)_5$.

*For any element* $r \in E_5$, *let* $\mathcal{S}(r)$ *be the support of* $r$*: that is, the set of all subsets* $\{i_1, i_2, \ldots, i_5\}$ *that index nonzero monomials* $e_{i_1} \cdots e_{i_5}$ *in* $r$. *Suppose* $r$ *is pure, that is* $r = a_1a_2 \cdots a_5$ *where each* $a_i \in E_1$. *Then the coefficients of the monomials that make up* $r$ *are the minors of a* $9 \times 5$ *matrix whose* $i$*th*

*column is given by $a_i$. In particular, then, the set $\mathcal{S}(r)$ is the set of bases of the matroid on the set [9] given by the rows of this matrix.*

*Consider a nonzero linear combination $r = \alpha p + \beta q$. We show that $r$ cannot be pure as follows. We can assume that $\alpha \neq 0$, by replacing $p$ and $q$ with $\tau p$ and $\tau q$, respectively, otherwise. Then put $B = \{1, 2, 6, 8, 9\}$. By writing out $r$ we find $B_1 \in \mathcal{S}(r)$. However, neither $B_1 \cup \{5\} \setminus \{9\}$ nor $B_1 \cup \{7\} \setminus \{9\}$ are in $\mathcal{S}(r)$. Thus $\mathcal{S}(r)$ does not satisfy the main axiom for the set of bases of a matroid. It follows that $(J^0)_5$ contains no pure elements.*

## 5. A Presentation of $I^0$

The goal of this section is to exhibit a presentation of $I^0$ as an $E$-module. Using the generating set $Z$ we only need to describe basic relations among elements of $Z$.

First we recall a little about the minimal resolution of the initial ideal $J = In(I^0)$. For that it is convenient to start with the monomial ideal $J'$ of the polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$ generated by the same monomials as $J$ (i.e., $e_{\bar{T}}$ where $T \in \mathbf{nbc}_\ell$) where $x_i$ is substituted for $e_i$ $(i = 1, \ldots, n)$. The ideal $J'$ is the Stanley-Reisner ideal of a simplicial complex $\Delta'$ that is the canonical Alexander dual to the $\mathbf{nbc}$-complex $\Delta$ (cf., for example, [6] and [15]). By the latter we mwan the complex on $[n]$ whose simplexes are the $\mathbf{nbc}$-sets. Keeping this in mind, an explicit realization of the minimal resolution of $J'$ (as an $\mathcal{R}$-module) can be given in terms of the homology of links of $\Delta$ or the local homology of the lattice of the least common multiples of $\mathbf{tbc}$-sets. These constructions can be deduced from [18] or [15]. Then a minimal resolution of $J$ as an $E$-module can be constructed using [1] or [6].

For a presentation of $J$ we do not need any of those general constructions. A generating set of $J$ is mentioned above. In order to describe basic relations among the generators denote by $\mathbf{nbc}'$ the subset of $\mathbf{nbc}_{\ell-1}$ consisting of sets $S$ lying each in at least two $\mathbf{nbc}$-bases. Notice that $\mathbf{nbc}'$ can be identified with the rank 2 part of the lattice of the least common multiples of $\mathbf{tbc}$-sets of cardinality $n - \ell$. Then put for every $S \in \mathbf{nbc}'$

$$N(S) = \{i \in [n] \setminus S | S \cup \{i\} \in \mathbf{nbc}_\ell\}$$

and denote by $i(S)$ the minimal element of $N(S)$. The natural basic relations among these generators can be divided in two kinds. Relations $r(S, i)$ of the first kind are parametrized by all the pairs $(S, i)$ with $S$ as above and $i \in N(S) \setminus \{i(S)\}$. The relation $r(S, i)$ is

$$\epsilon([n] \setminus S, i) e_i e_{\bar{S} \setminus \{i\}} - e_{i(S)} e_{\bar{S} \setminus \{i_S\}} = 0 \tag{5.1}$$

where $\epsilon(T, i) = |\{j \in T | j < i\}|$. The relations $s_{T,j}$ of the second kind are parametrized by pairs $(T, j)$ where $T \in \mathbf{nbc}_\ell$, $j \in T$, and the relation $s(T, j)$ is $e_j e_{\bar{T}} = 0$.

There is a couple of features of the relations of the first kind that we will use later. First, they are partitioned in parts parametrized by $\mathbf{nbc}'$ and the part corresponding to $S$ contains $|N(S)| - 1$ relations. Second, instead of

the particular relations (5.1) that we have chosen as basic, one can take any similar relations using a set of pairs $\{\{i,j\}|i,j \in N(S)\}$ as long the graph on $N(S)$ whose edges are these pairs is a tree containing all the elements of $N(S)$.

The minimal resolution of $J = In(I^0)$ gives some information about the minimal resolution of $I^0$. As it was observed in [6], $J$ is a flat degeneration of $I^0$. It also was proved in [6] that the minimal resolutions of $J$ and $I^0$ are linear whence these two resolutions have the same dimensions of the corresponding terms.

We already know that generating sets of $J$ and $I^0$ are both in one-to-one correspondences with $\mathbf{nbc}_\ell$.

The situation is harder though with linear relations (of the first kind) among $z(T)$. In general there are no sufficiently many relations among them involving only pairs (similar to (5.1)). Because of that we describe the basic relations in two stages. On the first stage, we establish linear relations between pairs of $z(F)$ that are indexed by flags more general than $\mathcal{F}(T)$ for $T \in \mathbf{nbc}_\ell$. We are able to find enough of them so they form a basis of the whole space of relations. However, these $z(F)$ are not in general linearly independent over $R$, and the second stage is occupied with expressing them in terms of the standard $z(T)$ that form an $R$-basis of $(I^0)_\ell$.

We call two maximal flags *close* if and only if they differ by at most one flat.

Let $S$ and $T$ be disjoint subsets of $[n]$, and set $s = \min S$, $t = \min T$. Let $\sigma$ be the shuffle of $S$ and $T$. Using (2.1) we have

$$\partial(e_S)(e_s - e_t)\partial(e_T) = \operatorname{sgn}(\sigma)\partial(e_{S\cup T}).$$

It follows that if $\pi$ is a partition of $[n]$ with $\pi = \{A_1, A_2, \dots, A_k\}$, $s = \min A_i$, and $t = \min A_{i+1}$, then

$$(e_s - e_t)z(\pi) = (-1)^{\sum_{j\le i}(|A_j|-1)} z(\tilde\pi), \tag{5.2}$$

where $\tilde\pi$ is the partition with $k-1$ parts, obtained from $\pi$ by joining $A_i$ with $A_{i+1}$.

Suppose that $F$ and $F'$ are distinct, close, maximal flags: that is, $F = (X_0, X_1, \dots, X_\ell)$ and $F'$ differs from $F$ by replacing $X_k$ with $X'_k$ for a single $k$, $0 < k < \ell$. Let $\tilde\pi$ be the partition obtained from $\pi(F)$ or (equivalently) $\pi(F')$ by joining the $k$th and $k+1$st parts. We find that $z(F)$ and $z(F')$ are both divisors of $z(\tilde\pi)$ using (5.2). In order to be more specific, let $a = \min X_k \setminus X_{k-1}$ and $b = \min X_{k+1} \setminus X_k$. Obtain $a'$ and $b'$ similarly using $X'_k$ instead of $X_k$. Then

**Lemma 5.1.** *In the notation above,*

$$(e_a - e_b)z(F) - (-1)^{|X_k| - |X'_k|}(e_{a'} - e_{b'})z(F') = 0. \tag{5.3}$$

*Proof.* Immediate from equation (5.2). ∎

Now we define the class of flags and monomials we will use. Recall that if the elements of the $\ell$-tuple $U = (u_1, u_2, \dots, u_\ell)$ comprise a base, $\mathcal{F}(U)$

denotes the flag whose flat in rank $p$ is the closure of the last $p$ elements of $U$. Previously we considered $U$ which were **nbc**-bases written in increasing order. Call such a $U$ and its flag $\mathcal{F}(U)$ *standard*. In what follows, we will make use of other orderings of **nbc**-bases.

Any maximal flag $F$ defines an ordered base, which we denote by $\varphi(F)$, the $\ell$-tuple whose $p$th element is equal to $\min(X_p \setminus X_{p-1})$, for $1 \leq p \leq \ell$. Clearly $\varphi(F)$ is an ordered base with $F = \mathcal{F}(\varphi(F))$. Recall from Lemma 3.2 that if $U$ is standard, then $\varphi(\mathcal{F}(U)) = U$. If $U$ is not in increasing order, however, $\varphi(\mathcal{F}(U))$ may not equal $U$. Using the similar definition from [4], call any ordered base $U$ *neat* if $\varphi(\mathcal{F}(U)) = U$.

**Lemma 5.2.** *A neat ordered base is* **nbc** *(but not necessarily increasing).*

*Proof.* If $U$ is increasing and neat, then $U$ is an ordered **nbc**-base. If $U = (u_1, u_2, \ldots, u_\ell)$ is neat but $u_{i-1} > u_i$ for some $i$, then we show that the ordered base $U'$ obtained by transposing $u = u_{i-1}$ and $v = u_i$ is also neat. From this the conclusion will follow by induction.

Let $X = cl\,\{u_{i+1}, \ldots, u_\ell\}$. Since $U$ is neat, $v = \min(X \vee \{v\} \setminus X)$ and $u = \min(X \vee \{v, u\} \setminus X \vee \{v\})$. The inclusion

$$X \vee \{u\} \setminus X \subset (X \vee \{v, u\}) \setminus (X \vee \{v\})$$

implies $u = \min X \vee \{u\} \setminus X$ and

$$\begin{aligned} \min(X \vee \{v, u\} \setminus X \vee \{u\}) &\geq \min(X \vee \{v, u\} \setminus X) \\ &= v. \end{aligned}$$

Since $v$ is in the set on the left, we have equality, and $U'$ is neat. □

Now fix $S \in \mathbf{nbc}'$. We shall construct a tree whose vertices are labeled by $N(S)$, and whose edges give $|N(S)| - 1$ linear relations of the type (5.3).

Write $S = \{s_1, s_2, \ldots, s_{\ell-1}\}$ with $s_1 < \cdots < s_{\ell-1}$. Some notation is needed for ordered **nbc**-bases obtained by adding an element of $N(S)$ to $S$: for $a \in N(S)$ and $0 \leq k \leq \ell$, put $S(a, k) = (s_1, \ldots, s_{k-1}, a, s_k, \ldots, s_{\ell-1})$. If $a > s_{k-1}$, that is, if $a$ appears at or to the left of its place in increasing order, call $S(a, k)$ *early*.

In what follows, set $X^k = cl\,\{s_k, s_{k+1}, \ldots, s_{\ell-1}\}$, for $1 \leq k \leq \ell - 1$. We define a graph $\Gamma = \Gamma(S)$ from $S$ and $N(S)$ as follows. The vertices of $\Gamma$ are those sets $S(a, k)$ which are neat and early. The edges are the pairs $S(a, k)$ and $S(b, k + 1)$ for which $a \vee X^k = b \vee X^k$. Note that, if two vertices of $\Gamma$ are joined by an edge, then the flags of those two vertices are close.

**Example 5.3.** *Let $M$ be the matroid of Example 4.5 (Figure 1(a)). Of the 14* **nbc***-bases, only* 125, 135, 145, *and* 157 *contain the set $S = \{1, 5\}$, so in this case $N(S) = \{2, 3, 4, 7\}$. One finds that the vertices of $\Gamma(S)$ are $S(2, 1)$, $S(a, 2)$ for each $a \in N(S)$, and $S(7, 3)$. $\Gamma$ is a tree, rooted at $S(2, 1) = (2, 1, 5)$, shown in Figure 2. (The ordered sets are written with the element of $N(S)$ emphasized.)*

*Similarly, taking $S = \{1, 3\}$ gives $N(S) = \{5, 6, 7, 8\}$, and the resulting tree is also shown in Figure 2.*

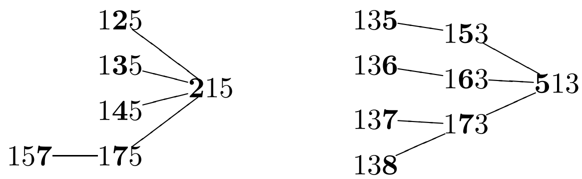



FIGURE 2. Trees $\Gamma(S)$ for $S = \{1,5\}$ and $S = \{1,3\}$

**Lemma 5.4.** *If $a \in N(S)$, $S(a,k)$ is early, and $a = i(S) = \min N(S) \cap (X^k \vee \{a\} \setminus X^k)$, then $S(a,k)$ is neat.*

*Proof.* Since $S(a,k)$ is early and $a \in N(S)$, $S(a,r)$ is standard for some $r \geq k$: let $F = \mathcal{F}(S(a,r))$. The first $\ell - r$ and last $k$ flats of $S(a,k)$ agree with those of $F$, so we only need check the condition in ranks $\ell - r + 1$ through $\ell - k + 2$. In rank $\ell - i$, for $i$ with $k \leq i < r$, we have $s_i = \min X^i \vee \{a\}$ since $X^i \vee \{a\}$ is a flat of $F$; therefore $s_i = \min X^i \setminus X^{i+1}$. In the remaining rank, $\ell - k + 2$, let $b = \min(X^k \vee \{a\}) \setminus X^k$. Then $S(b,k)$ is neat, so by Lemma 5.2, $b \in N(S)$. By hypothesis, then, $b = a$. □

As before, let $S \in \mathbf{nbc}'$ and $a = \min N(S)$.

**Proposition 5.5.** *$\Gamma(S)$ is a tree, rooted at $S(a,1)$. The leaves of $\Gamma(S)$ are the standard **nbc**-bases indexed by $N(S)$.*

*Proof.* We claim that any vertex $S(b,k+1)$ for $k \geq 1$ is connected to exactly one vertex $S(c,k)$. Since such an $S(c,k)$ must be neat, the only possibility would be

$$c = \min(N(S) \cap (X^k \vee \{b\} \setminus X^k)).$$

We have only to show that, for this choice of $c$, $S(c,k)$ is actually in $\Gamma$.

$S \cup \{b\}$ is a **nbc**-base; let $F = (Y_0 < Y_1 < \cdots < Y_\ell)$ be its standard flag. Since $S(b,k+1)$ is early, the last $k+1$ flats of $F$ and $\mathcal{F}(S(b,k+1))$ agree; in particular, $X^{k-1} \vee \{b\} = Y_{l-k+2}$. Since $F$ is a standard flag, $s_{k-1} = \min Y_{\ell-k+2}$, so $s_{k-1} < c$, which means $S(c,k)$ is early.

$S(c,k)$ is also neat, by Lemma 5.4, and so (by definition) it is a vertex of $\Gamma$. This proves the first claim.

To prove the second claim, observe that any $S(c,k)$ that is standard is certainly in $\Gamma$. Suppose that such a vertex $S(c,k)$ is connected to some $S(b,k+1)$ (hence is not a leaf). Then $c < s_k < b$. We have

$$\begin{aligned}
\min(X^k \vee \{c\} \setminus X^{k+1}) &= \min(X^k \vee \{b\} \setminus X^{k+1}) \\
&= \min\left(X^k \vee \{b\} \setminus X^{k+1} \vee \{b\}\right) \cup \left(X^{k+1} \vee \{b\} \setminus X^{k+1}\right) \\
&= \min\{s_k, b\} = s_k,
\end{aligned}$$

since $S(b,k+1)$ is neat. The same argument applied to $S(c,k)$ shows that the minimum is $c$, a contradiction.

Conversely, if $S(c,k)$ is not standard, then the argument of Lemma 5.2 shows that $S(c,k+1)$ is also in $\Gamma$. The two are connected, so $S(c,k)$ is not a leaf. $\square$

Given such a tree $\Gamma(S)$, construct a graph $\mathfrak{t} = \mathfrak{t}(S)$ with vertices $N(S)$ and an edge for a pair $(a,b)$, $a \neq b$, if and only if $S(a,k)$ and $S(b,k+1)$ are joined by an edge for some $k$ in $\Gamma(S)$. Then $\mathfrak{t}$ is also a tree, since its edges must have $a < b$. Its construction can also be described algorithmically: for each non-minimal element $b \in N(S)$, write the base $S \cup \{b\}$ in increasing order to give a standard ordered base $S(b,r)$ for some $r$. Now move $b$ to the left to get $S(b,k)$, for $k = r-1$, $k = r-2$, $\dots$, as long as $S(b,k)$ remains neat. For some greater $k$, $S(b,k)$ will not be neat. Then $\varphi(\mathcal{F}(S(b,k))) = S(a,k)$ for some smaller $a \in N(S)$. Connect $a$ to $b$, and repeat until the resulting graph is connected.

By construction, we have:

**Proposition 5.6.** *For each $S \in \mathbf{nbc}_{\ell-1}$, each edge of the graph $\mathfrak{t}(S)$ on the set $N(S)$, constructed above, gives a pair of close flags, $F = \mathcal{F}(S(a,k))$ and $F' = \mathcal{F}(S(b,k+1))$, for some $k$.*

Then Lemma 5.1 describes a linear relation between $z(F)$ and $z(F')$. These are the relations of the first kind. It is much easier to generalize the relations of the second kind. For each $T \in \mathbf{nbc}_\ell$, write $\pi(T) = \{A_1, \dots, A_\ell\}$. Then we have $n - \ell$ relations

$$(e_j - e_{\nu(A_i)})z(T) = 0 \tag{5.4}$$

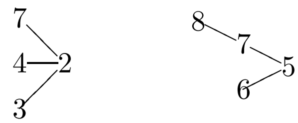
where $j \in A_i \setminus \{\nu(A_i)\}$, $i = 1, 2, \dots, \ell$.

What is left for us to observe is that all these relations, of the first and the second kind together, form an $R$-basis of the relation space.

**Theorem 5.7.** *The relations* (5.3) *where a pair $\{F, F'\}$ runs through all edges of all the graphs $\mathfrak{t}(S)$ ($S \in \mathbf{nbc}'$) together with the relations* (5.4) *where $T \in \mathbf{nbc}_\ell$ form an $R$-basis of the relation space.*

*Proof.* Observe that for each $S$ we have precisely $|N(S)| - 1$ relations of the first kind. Since this number coincides with the number of relations of the first kind for $J$ and the same is true for the relations of the second type, the total number of the relations is correct. Now for each of those relations we can obtain a relation for the generators of $J$ by taking the initial monomials of $z(F)$, $z(F')$, $z(T)$ and the coefficients. The latter relations are $R$-linearly independent. The nontrivial part of the reasoning for that is that the involving pairs for the relations of the first kind with fixed $S$ from $\mathbf{nbc}'$ form a tree. Thus the relations for the elements from $Z$ are also independent which completes the proof. $\square$

**Example 5.8.** *Continuing the previous Example 5.3, we obtain for* $S = \{1, 5\}$ *and* $S = \{1, 3\}$, *the following respective trees* $\mathfrak{t}(S)$*:*




*For example, the edge* $(5, 6)$ *of* $\mathfrak{t}(\{1, 3\})$ *comes from the edge joining* $S(6, 2)$ *to* $S(5, 1)$ *in Figure 2. The corresponding flags* $F = (\emptyset < \{3\} < \{3, 6\} < [8])$ *and* $F' = (\emptyset < \{3\} < \{1, 2, 3, 4\} < [8])$ *are close. By Lemma 5.1,*

$$(e_6 - e_2)z(F) - (e_1 - e_5)z(F') = 0.$$

The remaining step of writing a presentation of $I^0$ is to reexpress flags of elements of the trees $\Gamma$ in terms of standard flags. For that it is convenient to use linear dependencies (over $R$) that are essentially equivalent to the flag relations (2.1.1) of [16],

**Lemma 5.9.** *The elements* $z(F)$ *satisfy the equation*

$$\sum_{X_{i-1} \subset Y \subset X_{i+1}} z(X_0 \subset \cdots \subset X_{i-1} \subset Y \subset X_{i+1} \subset \cdots \subset X_\ell) = 0,$$

*for any maximal flag* $(X_0 \subset X_1 \subset \cdots \subset X_\ell)$ *and any* $i$, $0 < i < \ell$.

*Proof.* Let $A_1, A_2, \ldots, A_k$ denote the sets $Y \setminus X_{i-1}$, as $Y$ ranges over all flats between $X_{i-1}$ and $X_{i+1}$. Note that these sets partition $X_{i+1} \setminus X_{i-1}$, since $Y$ is the closure of $X_{i-1} \cup \{y\}$ for any $y \in Y \setminus X_{i-1}$. Now let $\sigma$ be a permutation of $[n]$ so that the elements of $X_{i+1} \setminus X_{i-1}$ are consecutive, and the elements $A_j$ come before $A_{j+1}$, for all $1 \le j < k$.

To prove the statement it is enough to factor out those terms that appear in each summand and show instead that

$$\sum_{j=1}^{k} \operatorname{sgn}(\sigma^j)\partial(e_{A_j})\partial(e_{A_1}e_{A_2}\cdots\widehat{e_{A_j}}\cdots e_{A_k}) = 0,$$

where $\sigma^j$ is the permutation obtained from $\sigma$ by moving the block $A_j$ in front of $A_1$. Put $a_j = |A_j|$ for each $j$. Since $\partial$ satisfies the Leibniz rule, the sum becomes

$$\begin{aligned} &\operatorname{sgn}(\sigma)\sum_{j=1}^{k}(-1)^{a_j \sum_{p<j} a_p}\partial(e_{A_j})\partial(e_{A_1}e_{A_2}\cdots\widehat{e_{A_j}}\cdots e_{A_k}) \\ &= \operatorname{sgn}(\sigma)\partial\left(\sum_{j=1}^{k}(-1)^{a_j \sum_{p<j} a_p}\partial(e_{A_j})e_{A_1}e_{A_2}\cdots\widehat{e_{A_j}}\cdots e_{A_k}\right) \\ &= \operatorname{sgn}(\sigma)\partial(\partial(e_{A_1}e_{A_2}\cdots e_{A_k})) \\ &= 0. \end{aligned}$$

$\square$

**Proposition 5.10.** *Let $S$ be an independent set with $|S| = \ell - 1$. If $S(a,k)$ is in $\Gamma(S)$, then*

$$z(S(a,k)) = \sum_{S(b,j)} (-1)^{j-k} z(S(b,j)),$$

*where the sum is taken over all standard $S(b,j)$ that are descendents of $S(a,k)$ in $\Gamma(S)$.*

*Proof.* Choose any $S(a,k)$ in $\Gamma$. If it is not standard, then we claim that $z(S(a,k)) = -\sum_b z(S(b,k+1))$, taking the sum over all $S(b,k+1)$ in $\Gamma$ connected to $S(a,k)$. By Lemma 5.9,

$$z(S(a,k)) = -\sum_i z(F_i)$$

where sum is taken over all flags $F_i$ that differ from $\mathcal{F}(S(a,k))$ in rank $\ell - k$, that is, agreeing at all flats except $X^k$. The image of $\varphi$ is always neat, so it follows from Lemma 5.2 that $\varphi(F_i) = S(b_i, k+1)$ for some $b_i \in N(S)$. By the neatness, $b_i = \min(X^{k+1} \vee \{b_i\} \setminus X^{k+1})$ and $a = \min(X^k \vee \{a\} \setminus X^k)$. The first set is contained in the second, so $b_i \geq a$. By assumption $S(a,k)$ is not standard: $a > s_k$. This means $S(b_i, k+1)$ is early, hence an element of $\Gamma$, and the claim is proven.

The proposition now follows by induction. □

By combining Proposition 5.6 with Proposition 5.10, our presentation is described completely.

**Example 5.11.** *Continuing the previous example, the same relation becomes*

$$(e_6 - e_2)(-z(136)) - (e_1 - e_5)(z(135) + z(136) + z(137) + z(138)) = 0$$

*in terms of the standard basis for $I^0$.*

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